

Trigonometric Approximation in Sobolev-Grand Lebesgue Spaces.

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Abstract.

We study in this short preprint the theory of trigonometric approximation in the so-called Banach functional rearrangement invariant Sobolev-Grand Lebesgue Spaces.

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1 Introduction. Notations. Statement of problem.

Let $X = [0, 2\pi]$ with *normed* Lebesgue measure $d\mu(x) = \mu(dx) = dx/(2\pi)$ be a classical probability space and let B be Banach rearrangement invariant real valued functional space builded over X , equipped with norm $\|f\|_B$, consisting on the 2π – periodical functions.

The space B is called a *homogeneous Banach space*, (abbreviated HBS), on the set X , see [18], [10], [29] etc., iff:

(a). It is linear subspace of $L_1 = L_1(X, \mu)$, such that

$$(a). \quad \exists C \in (0, \infty) \Rightarrow \|f\|_1 \leq C \|f\|_B, \quad (1.1)$$

(b). The translation $U_t[f]$, $U_t[f](x) := f(x - t)$ is continuous isometry of B onto itself, namely:

$$(b). \quad \forall f \in B \Rightarrow \|U_t f\|B = \|f\|B \quad (1.2)$$

and (c).

$$(c). \quad \lim_{t \rightarrow 0} \|U_t[f] - f\|B = 0. \quad (1.3)$$

For instance, the classical Lebesgue - Riesz spaces $L_p = L(p) = L_p(X, \mu)$, $1 \leq p < \infty$ with ordinary norms

$$\|f\|_p = |f|L_p = |f|L(p) = |f|L_p(X, \mu) = \left[\int_X |f(x)|^p \mu(dx) \right]^{1/p}$$

are HBS.

Define for each such a function $f = f(x)$, $f : X \rightarrow R$ from this space $B : f \in B$ its B - module of continuity

$$\omega_B[f](\delta) \stackrel{def}{=} \sup_{h:|h| \leq \delta} \|U_h[f] - f\|B, \quad \delta \in X. \quad (1.4)$$

We admit in (1.4) that $f(x) = 0$ if $x \notin X$, and will write for brevity

$$\omega[f](\delta)_p = \omega_{L(p)}[f](\delta) = \sup_{h:|h| \leq \delta} |U_h[f] - f|_p, \quad \delta \in X. \quad (1.4a)$$

Evidently,

$$B \in HBS, \quad f \in B \Rightarrow \lim_{\delta \rightarrow 0+} \omega_B[f](\delta) = 0.$$

Note that in the approximation theory may be successfully used another modules of continuity, see [10]. The case of general rearrangement spaces B is studied in [4], [21], [31], [32].

The purpose of this note is to extend the characterization of best trigonometric approximation by the module $\omega_B[f](\delta)$ to the suitable subspace of the so-called Banach space of periodic functions, namely, Grand Lebesgue Spaces (GLS), as well as to the so-called exponential Orlicz spaces.

The case when the space B coincides with some Sobolev space W_p^r , $p \geq 1$, $r = 0, 1, 2, \dots$ will be also considered further.

Hereafter C, C_j will denote any non-essential finite positive constants.

Further, let $(Y, \|\cdot\|Y)$ be any rearrangement invariant (r.i.) space on the set X ; denote by $\phi(Y, \delta)$ its fundamental function

$$\phi(Y, \delta) = \sup_{A, \mu(A) \leq \delta} \|I(A)\|Y, \quad (1.5)$$

where as usually $I(A)$ denotes the ordinary indicator function of the measurable set A :

$$I(A) = I(A, x) = 1, x \in A; \quad I(A) = I(A, x) = 0, \quad x \notin A.$$

Denote by $T(n)$, $n = 1, 2, \dots$ the set (subspace) of all the trigonometric polynomials on the variable x ; $x \in X$ of degree less or equal n with coefficients from the space B and define correspondingly for the space B , in particular, for the space $G(\psi)$, and for each function f from this space the minimal error of its trigonometrical approximation

$$E_n[f]B \stackrel{\text{def}}{=} \inf_{g \in T(n)} \|f - g\|B. \quad (1.6)$$

We have to take as above in (1.6) for brevity

$$E_n[f]_p \stackrel{\text{def}}{=} E_n[f]L_p = \inf_{g \in T(n)} \|f - g\|_p. \quad (1.7)$$

Definition 1.1. The function f from the Banach space B is said to be *trigonometric approximated* in this space, write: $f \in TA(B)$, if

$$\lim_{n \rightarrow \infty} E_n[f]B = 0. \quad (1.8)$$

2 The case of Grand Lebesgue Spaces.

We recall first of all some needed facts about Grand Lebesgue Spaces (GLS).

Recently, see [5], [11], [12], [15]-[17], [20], [22]-[27] etc. appear the so-called Grand Lebesgue Spaces $GLS = G(\psi) = G(\psi; b)$, $b = \text{const} \in (1, \infty]$ spaces consisting on all the measurable functions $f : X \rightarrow R$ with finite norms

$$\|f\|G(\psi) = G(\psi, b) \stackrel{\text{def}}{=} \sup_{p \in [1, b)} \left[\frac{|f|_p}{\psi(p)} \right]. \quad (2.1)$$

Here $\psi(\cdot)$ is some continuous positive on the semi - open interval $[1, b)$ function such that

$$\inf_{p \in [1, b)} \psi(p) > 0.$$

It is evident that $G(\psi; b)$ is Banach functional rearrangement invariant (r.i.) space and $\text{supp}(G(\psi_b)) := \text{supp } \psi_b = [1, b)$.

Let the *family* of measurable functions $h_\alpha = h_\alpha(x)$, $x \in X$, $\alpha \in A$, where A be arbitrary set, be such that

$$\exists b \in (1, \infty], \quad \forall p \in [1, b) \Rightarrow \psi^A(p) := \sup_{\alpha \in A} |h_\alpha|_p < \infty.$$

Such a function $\psi^A(p)$ is named as a *natural function* for the family A . Obviously,

$$\sup_{\alpha \in A} \|h_\alpha\| G\psi^A = 1.$$

Note that the case when $\sup_{p \in [1, b)} \psi(p) < \infty$ is trivial for us; if for instance $b < \infty$ and $\psi(b-0) < \infty$, then the space $G(\psi, b)$ coincides with ordinary Lebesgue-Riesz space $L_b(X)$. Therefore, we can and will suppose in the sequel without loss of generality

$$\sup_{p \in [1, b)} \psi(p) = \lim_{p \rightarrow b-0} \psi(p) = \infty. \quad (2.2)$$

These spaces are used, for example, in the theory of probability, theory of PDE, functional analysis theory of Fourier series, theory of martingales etc.

The problem of trigonometric approximation in the Grand Lebesgue Spaces is considered in more complicated terms in the article [7].

The spaces $G(\psi; b)$ are non-separable, but they satisfy the Fatou property. As long as its Boyds indices γ_-, γ_+ are correspondingly

$$\gamma_- = 1/b, \quad \gamma_+ = 1,$$

we conclude that the spaces $G(\psi; b)$ are interpolation spaces not only between the spaces $[L_1, L_\infty]$ but between also the spaces $[L_1, L_s]$ for all values s for which $s > b$.

Note that an another approach to the problem of combine the L_p spaces may be found in the articles [2], [3], [8], [30].

Let us introduce an important subspace of the whole space $G(\psi; b)$.

Definition 2.1. The (closed) subspace $G^o(\psi) = G^o(\psi; b)$ of the whole GLS $G\psi : G^o(\psi) \subset G(\psi)$ consists by definition on all the functions $\{f\}$ from the whole space $G(\psi; b)$ for which

$$\lim_{p \rightarrow b-0} \left\{ \frac{|f|_p}{\psi(p)} \right\} = 0. \quad (2.3)$$

Of course, the functions belonging to the space $G^o(\psi) = G^o(\psi; b)$ have at the same norm (2.1) as in the space $G(\psi) = G(\psi; b)$.

It is known, see [26], [43], that the spaces $G^o(\psi) = G^o(\psi; b)$ have absolute continuous norm and coincides with closure of the set of all bounded measurable functions. Alike in the theory of Orlicz spaces, they are reflexive and separable.

Example 2.1. Let the numerical valued *random variable* (*r.v.*) (measurable function) $\xi = \xi(x)$, $x \in X$ be defined on our probability space and has a standard Gaussian (normal) distribution. Then it belongs to the GLS $G\psi_{1/2}$, where for each $m = \text{const} \in (0, \infty)$ we define

$$\psi_m(p) := p^{1/m}, \quad p \geq 1.$$

Indeed, it is easily to calculate

$$|\xi|_p \asymp p^{1/2}, \quad p \geq 1.$$

Therefore, $\xi \in G\psi_{1/2}$, but $\xi \notin G^o\psi_{1/2}$.

On the other hand,

$$\forall m \in (0, 2) \Rightarrow \xi \in G^o\psi_{1/m}.$$

Let us now consider a generalizations of the classical results of trigonometric approximation into the Grand Lebesgue Spaces.

Theorem 2.1. Suppose the function f belongs to the space $G\psi$. Statement: this function is trigonometric approximated in this space, $f \in TA(G\psi)$, if and only if it belongs to the subspace $G^o(\psi) = G^o(\psi; b)$.

Furthermore, in this case

$$E_n[f]G\psi \leq C_1(G\psi) \omega_{G\psi}[f](2\pi/n), \quad n = 1, 2, \dots; \quad (2.4)$$

and conversely

$$\omega_{G\psi}[f](2\pi/n) \leq C_2 n^{-1} \sum_{k=1}^n E_k[f]G\psi. \quad (2.5)$$

Proof.

1. Assume at first $f \in G^o(\psi)$, then the function $f(\cdot)$ has an Absolute Continuous Norm (ACN) in the space $G\psi$ or equally in the space $G^o\psi$, see [22], [24], [25], [26]. Following, $G^o(\psi)$ is homogeneous Banach space (HBS), see [4], chapter 1, section 3; and hence

$$\lim_{n \rightarrow \infty} \omega_{G\psi}[f](2\pi/n) = 0. \quad (2.6)$$

It remains to apply the classical results about trigonometric approximation in these spaces, see [10], [18], [29] etc. to deduce the equalities (2.4) and (2.5).

Note in addition to this pilcrow that in (2.4) the apparatus for correspondent approximation can be the convolution with the classical trigonometric kernels: Fejer F_n , Jackson J_n , Fourier S_n , Riesz R_n , Dirichlet D_n , Vallee Poussin V_n etc.

2. Two examples of the direct estimates. Assume $f \in G^o(\psi)$.

Let us write the famous Jackson's proposition for the L_p , $p \geq 1$ spaces:

$$\|f - J_n * f\|_p \leq C_1 \omega[f](2\pi/n)_p. \quad (2.7)$$

It is important to note that the constant C_1 one can choose not depending on the variable p . Furthermore, the estimate (2.7) there holds still for the value $p = \infty$, where C_1 may be taken such that $C_1 = 3$.

As long as $f \in G\psi$,

$$\omega[f](2\pi/n)_p \leq \omega_{G\psi}[f](2\pi/n) \psi(p), \quad p \in (1, b),$$

and we get after substituting into (2.7)

$$|f - J_n * f|_p \leq C_1 \omega_{G\psi}[f](2\pi/n) \psi(p),$$

or equally on the basis of the direct definition of $G\psi$ norm

$$\|f - J_n * f\|_{G\psi} \leq C_1 \omega_{G\psi}[f](2\pi/n),$$

which coincides with (2.4); the relation (2.5) may be grounded analogously.

A second example. Let for definiteness the number n be even number, and let $f \in G^o\psi = G^o\psi_b, b = \text{const} \in (1, \infty]$. One can use the-known Vallee-Poussin inequality

$$|f - V_n * f|_p \leq C_2 E_{n/2}[f]_p, \quad 1 \leq p < b.$$

Since

$$E_{n/2}[f]_p = \sup_{g \in T(n/2)} |f - g|_p \leq \sup_{g \in T(n/2)} \|f - g\|_{G\psi} \cdot \psi(p),$$

we deduce for any function f from the space $G^o\psi$

$$\|f - V_n * f\|_{G\psi} \leq C_2 \sup_{g \in T(n/2)} \|f - g\|_{G\psi} = E_{n/2}[f]_{G\psi}. \quad (2.8)$$

Note that the last expression tends to zero as $n \rightarrow \infty$ as long as $f \in G^o\psi$.

3. We continue. Suppose for certain function f from the space $G\psi$

$$\lim_{n \rightarrow \infty} E_n[f]_{G\psi} = 0.$$

Since the trigonometric system consists only on bounded functions, the last equality implies that the function f belongs to the closure in $G\psi$ norm of the set of all bounded measurable functions $G^{(b)}G\psi$. But the last space coincides with the space $G^o\psi$.

The rest follows from the theory of r.i. spaces, see the classical book of C.Bennet, R.Sharpley [4], chapter 1.

Definition 2.1. Let $G\psi$ and $G\nu$ be two Grand Lebesgue Spaces with at the same support of the generating functions ψ and ν . We will write $\nu \ll \psi$, or equally $G\nu \ll G\psi$, iff

$$\lim_{p \rightarrow b-0} \frac{\psi(p)}{\nu(p)} = 0. \quad (2.9)$$

Corollary 2.1. Let $0 \neq f \in G\psi \setminus G^o\psi$ and let $\nu = \nu(p)$ be arbitrary another Ψ function with at the same support as the source function $\psi = \psi(p)$, and such

that $G\nu \ll G\psi$. Then the function $f(\cdot)$ is not trigonometric approximated in the space $G\psi : f \notin TA(G\psi)$ but it is trigonometrical approximated in the space $G\nu : f \in TA(G\nu)$.

3 Main result: Approximation in Sobolev-Grand Lebesgue Spaces.

Denote as ordinary by W_p^r , $1 \leq p < \infty$, $r = 1, 2, \dots$ the classical Sobolev's space on the unit circle $X = [0, 2\pi]$ consisting on the 2π periodical functions $\{f\}$, in particular $f(0) = f(2\pi)$.

The space W_p^0 coincides with the classical Lebesgue - Riesz space $L_p(X)$.

The norm of a function f in this space W_p^r may be defined for instance as follows:

$$\|f\|_{W_p^r} \stackrel{def}{=} \left[|f|_p^p + |f^{(r)}|_p^p \right]^{1/p}. \quad (3.1)$$

Let also $\psi = \psi(p)$, $1 \leq p < b$, $b = \text{const} \in (1, \infty]$ be the ordinary Ψ - function.

Definition 3.1. The Sobolev-Grand Lebesgue Space $GW_r\psi$.

This space consists by definition from all the measurable functions having finite norm

$$\|f\|_{GW_r\psi} \stackrel{def}{=} \sup_{p \in (1, b)} \left\{ \frac{\|f\|_{W_p^r}}{\psi(p)} \right\}, \quad b = \text{const} \in (1, \infty]. \quad (3.2)$$

Define the following sequence of the Ψ - functions:

$$\theta_n(q) \stackrel{def}{=} n^{-1/q}, \quad q \in (s(1), s(2)),$$

$$s(1) = \text{const} \in (b, \infty), \quad s(1) < s(2) = \text{const} \leq \infty, \quad (3.3)$$

but the norm in this $G\theta_n = G\theta_n(s(1), s(2))$ space, more precisely, the sequence of these norms, is defined as follows

$$\|f\|_{G\theta_n} = \|f\|_{G\theta_n(s(1), s(2))} \stackrel{def}{=} \sup_{q \in (s(1), s(2))} \left[\frac{|f|_q}{\theta_n(q)} \right]. \quad (3.4)$$

Denote also for brevity

$$\Delta_n[f] = \Delta_n[f](x) := f(x) - J_n * f(x).$$

Theorem 3.1.

$$\| \Delta_n[f] \|_{G\theta_n(s(1), s(2))} \leq C_3(r) n^{-r} \frac{\|f\|_{GW_r\psi}}{\phi(G\psi, 1/n)}. \quad (3.5)$$

Proof. Suppose without loss of generality $\|f\|_{GW_r\psi} = 1$, considering the value of r to be fixed; then

$$\|f\|_{W_p^r} \leq \psi(p), \quad 1 \leq p < b.$$

We start from the well-known inequality, [1], [10], [13], [18] etc.:

$$\| \Delta_n[f] \|_q \leq C_3(r) n^{-r} n^{1/p-1/q} \|f\|_{W_p^r}, \quad 1 \leq p < q, \quad (3.6)$$

which may be transformed as

$$\begin{aligned} \frac{\| \Delta_n[f] \|_q}{n^{-1/q}} &\leq C_3(r) n^{-r} \frac{\|f\|_{W_p^r}}{n^{-1/p}} \leq C_3(r) n^{-r} \frac{\psi(p)}{n^{-1/p}} = \\ &= \frac{C_3(r) n^{-r}}{n^{-1/p}/\psi(p)} = \frac{C_3(r) n^{-r} \|f\|_{GW_r\psi}}{n^{-1/p}/\psi(p)}, \quad 1 \leq p < b < q \in (s(1), s(2)). \end{aligned} \quad (3.7)$$

We get taking the minimum over p :

$$\begin{aligned} \frac{\| \Delta_n[f] \|_q}{n^{-1/q}} &\leq C_3(r) n^{-r} \inf_{p \in (1, b)} \frac{\|f\|_{GW_r\psi}}{n^{-1/p}/\psi(p)} = \\ &= C_3(r) n^{-r} \frac{\|f\|_{GW_r\psi}}{\sup_p [n^{-1/p}/\psi(p)]} = C_3(r) n^{-r} \frac{\|f\|_{GW_r\psi}}{\phi(G\psi, 1/n)}, \end{aligned} \quad (3.8)$$

and taking further the maximum over q :

$$\| \Delta_n[f] \|_{G\theta_n(s(1), s(2))} \leq C_3(r) n^{-r} \frac{\|f\|_{GW_r\psi}}{\phi(G\psi, 1/n)},$$

Q.E.D.

Note that the obtained statement may be interpreted as some estimate for Kolmogorov widths of unit balls in Sobolev-Grand Lebesgue Space, cf.[6], [9], [23], [13], [14], [19], [28], [33], [34] etc.

4 Trigonometric approximation in Orlicz Spaces.

Denote by $L(N)$, where $N = N(u)$, $u \in R$ is certain Young-Orlicz function such that

$$N(u) \sim u^2, \quad u \in [-1, 1] \quad (4.1)$$

the Orlicz function space over source space X with the Luxemburg norm $\|f\|_{L(N)}$, $f : X \rightarrow R$.

The approximation problems by trigonometric polynomials in Orlicz spaces were investigated by several authors, see, for example, articles [35]-[42], where was considered as a rule the case when the generating function $N(u)$ satisfies the Δ_2 condition.

Recall that the Δ_2 condition means the separability and reflexivity of correspondent Orlicz space.

Let us note first of all that the so-called *exponential* Orlicz space $L(N)$ coincides up to norm equivalence with suitable Grand Lebesgue space $G\psi_N$, which admit us to obtain in turn the trigonometric approximation theorems in exponential Orlicz space.

Let $\psi = \psi(p)$, $p \in [1, b)$, $b = \text{const}$, $1 < b \leq \infty$ (or $p \in [1, b]$) be again bounded from below: $\inf \psi(p) > 1$ continuous inside the *semi-open* interval $[1, b)$ numerical function. We can and will suppose

$$b = \sup\{p, \psi(p) < \infty\},$$

so that $\text{supp } \psi = [1, b)$ or $\text{supp } \psi = [1, b]$. The set of all such a functions will be denoted by $\Psi(b)$, and we denote for brevity $\Psi := \Psi(\infty)$.

Suppose $b = \infty$ and $0 < \|f\| := \|f\|_{G\psi} < \infty$. Define the function

$$\nu(p) = \nu_\psi(p) = p \ln \psi(p), \quad 1 \leq p < b. \quad (4.2)$$

Recall that the Young - Fenchel, or Legendre transform $f^*(y)$ for arbitrary function $f : R \rightarrow R$ is defined (in the one-dimensional case) as follows

$$f^*(y) \stackrel{\text{def}}{=} \sup_x (xy - f(x)).$$

If the function $f(\cdot)$ is continuous and convex, then

$$f^{**}(x) = f(x),$$

theorem of Fenchel-Morau.

The so - called tail function $T_\zeta(y)$, $\zeta : X \rightarrow R$, $y \geq 0$ is defined by the formula

$$T_\zeta(y) \stackrel{\text{def}}{=} \max \{ \mu(x : f(x) > y), \mu(x : f(x) < -y) \}, \quad y \geq 0.$$

It is known in this case, i.e. when $b = \infty$ that

$$T_\zeta(y) \leq \exp \left(-\nu_\psi^*(\ln(y/\|\zeta\|)) \right), \quad y > e \cdot \|\zeta\|. \quad (4.3)$$

Conversely, if (4.3) there holds in the following version:

$$T_\zeta(y) \leq \exp \left(-\nu_\psi^*(\ln(y/K)) \right), \quad y > e \cdot K, \quad K = \text{const} > 0,$$

and the function $\nu_\zeta(p)$, $1 \leq p < \infty$ is positive, continuous, convex and such that

$$\lim_{p \rightarrow \infty} \psi(p) = \infty,$$

then $\zeta \in G\psi$ and besides

$$\|\zeta\|_{G\psi} \leq C(\psi) \cdot K. \quad (4.4)$$

Moreover, let us introduce the *exponential* Orlicz space $L(M)$ over the source probability space (Ω, F, \mathbf{P}) with proper Young-Orlicz function

$$M(u) = M_\psi(u) := \exp\left(\nu_\psi^*(\ln |u|)\right), \quad |u| > e$$

and as ordinary $M(u) = M_\psi(u) = \exp(C u^2) - 1$, $|u| \leq e$. It is known [43] that the $G\psi$ norm of arbitrary measurable function (r.v.) $\zeta = \zeta(x)$ is quite equivalent to the its norm in Orlicz space $L(M)$:

$$\|\zeta\|_{G\psi} \leq C_1 \|\zeta\|_{L(M)} \leq C_2 \|\zeta\|_{G\psi}, \quad 1 \leq C_1 \leq C_2 < \infty. \quad (4.5)$$

Evidently, this exponential Young-Orlicz function does not satisfy the Δ_2 condition.

We consider now the inverse problem. Namely, let the *exponential* Young-Orlicz $M = M(u)$ be a given. The "exponentiality" implies by definition that the function

$$\theta(z) = \theta_M(z) := \ln M(\exp z)$$

is continuous and convex. Then the correspondent generating function for equivalent Grand Lebesgue Space $\psi_M(p)$ may be builded by virtue of theorem Fenchel-Morau: $f^{**} = f$, $f'' > 0$, by the formula

$$\psi_M(p) = \exp\left(\frac{\theta_M^*(p)}{p}\right), \quad p \geq 1. \quad (4.6)$$

Indeed, we have

$$\exp\left(\nu_\psi^*(\ln |u|)\right) = M(u) = M_\psi(u), \quad |u| > e,$$

$$\nu_\psi^*(\ln |u|) = \ln M_\psi(u), \quad |u| > e,$$

therefore

$$\nu_\psi(\ln |u|) = [\ln M_\psi(u)]^* = \theta_M^*(u),$$

which implies (4.6).

Example 4.1. The estimate for the r.v. ξ of a form

$$|\xi|_p \leq C_1 p^{1/m} \ln^r p, \quad p \geq 2,$$

where $C_1 = \text{const} > 0$, $m = \text{const} > 0$, $r = \text{const}$, is quite equivalent to the following tail estimate

$$T_\xi(x) \leq \exp \left\{ -C_2(C_1, m, r) x^m \log^{-mr} x \right\}, \quad x > e.$$

as well as is equivalent to the belongings $\xi(\cdot)$ to the exponential Orlicz function with correspondent generating function $N(u) = N_{m,r}(u)$ of the form

$$N_{m,r}(u) = \exp \left(C_4(C_3, m, r) u^m \log^{-mr} u \right), \quad u > e.$$

It is important to note that the inequality (4.5) may be applied still when the r.v. ξ does not have the exponential moment, i.e. does not satisfy the famous Kramer's condition. Namely, let us consider next example.

Example 4.2. Define the following Ψ – function.

$$\psi_{[\beta]}(p) := \exp \left(C_3 p^\beta \right), \quad p \in [1, \infty), \quad \beta = \text{const} > 0.$$

The r.v. ξ belongs to the space $G\psi_{[\beta]}$ if and only if

$$T_\xi(x) \leq \exp \left(-C_4(C_3, \beta) [\ln(1+x)]^{1+1/\beta} \right), \quad x \geq 0.$$

as well as iff it belongs to the exponential Orlicz function with correspondent generating function $N(u) = N^{(\beta)}(u)$ of the form

$$N^{(\beta)}(u) = \exp \left(C_5(C_3, \beta) \ln(1+u)^{1+1/\beta} \right), \quad u > 1.$$

See also [20], [24].

Let us return to the source problem.

Theorem 4.1. Suppose the measurable function $f : X \rightarrow R$ belongs to certain exponential Orlicz space $L(M)$. We assert that this function is trigonometric approximated in this space: $f \in TA(L(M))$ if and only if

$$f \in G^o\psi_M(\cdot), \tag{4.7}$$

or equally

$$\lim_{\psi_M(p) \rightarrow \infty} \left\{ \frac{|f|_p}{\psi_M(p)} \right\} = 0. \tag{4.7a}$$

Proof. Since the GLS and correspondent exponential Orlicz norms are equivalent, the problems of trigonometric approximations in both the considered spaces are also equal. Our proposition follows immediately from Theorem 2.1.

We recall now the following definition about comparison of Orlicz spaces.

Definition 4.1. Let $L(N)$ and $L(K)$ be two Orlicz spaces with Young - Orlicz functions correspondingly $N = N(u)$, $K = K(u)$, $u \geq 0$. We will write $K \ll N$ or equally $L(K) \ll L(N)$, iff

$$\forall \lambda > 0 \Rightarrow \lim_{u \rightarrow \infty} \frac{K(\lambda u)}{N(u)} = 0. \quad (4.8)$$

Cf. the definition 2.1 (2.9).

Corollary 4.1. Suppose the measurable function $f : X \rightarrow R$ belongs to certain exponential Orlicz space $L(M)$. We assert that this function is trigonometric approximated in this space: $f \in TA(L(M))$ if and only if it belongs to some Orlicz space $L(K) : f \in L(K)$ such that $K \ll M$.

5 Concluding remarks.

The multidimensional case $X = [0, 2\pi]^d$, with or without weight, as well as the case of trigonometric approximation on the whole space $R = R^d$, may be investigated quite analogously, as well as the problem of algebraic approximation [1], [9], [23], [33].

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